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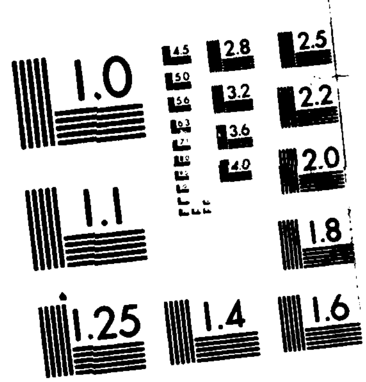

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U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
Software Analysis and Management System

Wild Bearing Analysis

EAAF

Technical Memorandum No. 3

July 10, 1985

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is one of a series of algorithm analysis reports performed for the US Army Intelligence Center and School covering selected algorithms in existing or planned Intelligence and Electronic Warfare (IEW) systems. This report addresses derivation of a method of accounting probabilistically for the possibility of a given bearing being wild. It is an interim report on one of several methodological investigations in progress in this area.

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PREFACE

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MARC MATHEMATICAL ANALYSIS RESEARCH CORPORATION
No 28

TO: Jim Gillis

SUBJECT: Wild Bearings

DATE: 10 July 1985

WILD BEARINGS

INTRODUCTION

A line of bearing may be erroneously attributed to a particular emitter. These bearings are referred to as wild bearings.

Several algorithms to estimate the location of a particular emitter and give an estimated error ellipse (EEP) utilizing lines of bearing (LOB's) to the emitter have been proposed in the literature. The possibility that some of the bearings are misidentified (wild) has not been taken into account in the derivations of the fix algorithms seen to date by MARC. Accounting for this possibility requires

(1) Estimation of the probability that a particular bearing is wild.

(2) Incorporation into a fix algorithm of the estimate of the probability that a particular bearing is wild.

It is the purpose of this report to show how this possibility may be taken into account by incorporating the probability of a bearing's being mis-identified (i.e. being a wild bearing) into fix algorithms. The methods of algorithm modification suggested here have two related properties that seem particularly desirable:

(1) Increasing probability of misidentification increases the size of the error ellipses.

(2) Bearings which are more likely to be correctly identified are weighted more.

The body of this report is divided into sections, as follows:

Section I is a brief discussion of considerations that affect the probability of a particular bearing's being wild. Section I also contains a reference to a paper discussing similar problems in the context of radar tracking algorithms.

Section II illustrates two approaches to using the probability of a wild bearing. The first uses an analogy with the ellipse combination methodology to illustrate the qualitative impact that should be expected from wild bearings. A method of modifying likelihood formulations is suggested by this analogy. A second approach is also shown which uses a weighted average of estimated locations. These two approaches are then compared.

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Sections III and IV show examples of these modifications for the sequential and perpendicular fix algorithms respectively.

Section V is related to Section I, and presents some of the mathematics of determination of some of the factors affecting the probability of a particular bearing's being wild.

Finally there are appendices in which notation is defined and various technical issues (such as the mathematical details of converting an X-Y covariance matrix into a weight on line of bearings) are addressed.

The materials presented here are merely meant to give an introduction to the problem of wild bearings.

I. PROBABILITY OF A PARTICULAR BEARING BEING WILD

Various factors might be considered when estimating the probability of a bearing's being wild. Analysis might involve:

(1) Number of emitters within the angular sector (confidence region) centered about the observed line of bearing (LOB). This number might be determinable but is more likely to be an estimate based perhaps on:
  (a) Emitter density in the zone of interest.
  (b) Area of the zone of interest lying in the angular sector.

(2) Quality (confidence value) of the emitter identification. There are several considerations that might be important:
  (a) Quality of match numbers generated in the emitter identification algorithm.
  (b) Comparative likelihood between the identified emitter and other candidates.
  (c) If an estimate of target location already exists then statistical distance between the observed and expected lines of bearing is a useful indicator of whether the new observation merits consideration.

The idea of mixing of various likelihoods and probabilities to determine the probability of correct identification has already been used in the context of radar tracking. See the work of Bar-Shalom. (For example, "Tracking In A Cluttered Environment With Probabilistic Data Association" Proc. 4th Symposium Nonliner Estimation, 1973 pp. 13-22). Because (2c) above requires the least assumptions about the emitter identification algorithm, we use it to illustrate how likelihood can be used.

Assume that considerations such as those in (1) above have generated a probability, P_W, of an identified bearing's being wild. The philosophy behind (2c) above is that further adjustments are necessary because a bearing that is far from previous observations is more likely to be wild than one that is close to previous observations. The 'usual' procedure is to assume that the probability P_W is modified proportional to its likelihood.

(i) Likelihood is based on a statistical distance between the old emitter location estimate and the new bearing.

(ii) Statistical distance is based on two forms of uncertainty. These are the uncertainty in the estimated location of the emitter and the uncertainty in the new bearing. The details depend on the forms in in which the normal distributions used by the fix algorithm are expressed. Technical details are discussed in section V.

(iii) To compute the proportionality it is first necessary to compute the average likelihood in the region in which LOBs can be accepted for this emitter. (This is often done by integrating over this region and then dividing by the area of the region.)

(iv) If P_M is the modified probability of being wild (after accounting for fix considerations) then, according to the likelihood approach

$$P_M:1-P_M \quad \text{as} \quad P_W*(\text{average likelihood}):(1-P_W)*(\text{observed likelihood})$$

The details of this analysis are completed in Section V. If P_W can be estimated and only an average effect of misidentification is desired then P_W may be used directly saving a degree of computational complexity.

II. USING THE PROBABILITY OF A BEARING BEING WILD IN ESTIMATES AND EEP's

A. Approach 1- Modify The Theory Behind the Various Fix Algorithms

Fix algorithms examined to date are derived by minimization of a quadratic form related to the log of the likelihood. (In this memo that quadratic form will also be referred to as statistical distance.) The modification to the quadratic form may be inferred by reference to likelihood reasoning, but the same analysis is somewhat easier to state in terms of the ellipse combination algorithm. This reasoning is presented below. The conclusion is that covariance matrices should be divided by the probability that the data are not 'wild.'

Compensating for a probability of wild ellipses in the combination algorithm is straightforward. Furthermore the result suggests a way of thinking of wild observations which is applicable to the least squares formulations used to derive the standard fix techniques. The following properties of the combination algorithm will be the basis for the analysis:

(1) The combination is independent of order.

(2) If n identical ellipses are combined then the resultant ellipse has a covariance matrx = 1/n times the original.

Now imagine that a fraction P of the ellipses are expected to be wild. The correct result will be obtained if the resultant covariance matrix is multiplied by 1/(1-P). Since combination is independent of order the same result would apply regardless of the number of individual covariance matrices mixed in.

Thus given a theory which is based on minimizing a least squares function of emitter location (X,Y) (equivalent to maximizing the normal likelihood function) such as the minimization of squared angular error

$$J(X,Y)=\sum_{k=1}^{n}[\tilde{\theta}_k-\theta_k(X,Y)]^T\psi_{\theta k}^{-1}[\tilde{\theta}_k-\theta_k(X,Y)]$$

there is a simple modification to account for an estimate of the probability of being a wild bearing. If P_k is the probability of the kth bearing being wild then the modified version is

$$J_P(X,Y)=\sum_{k=1}^{n}(1-P_k)[\tilde{\theta}_k-\theta_k(X,Y)]^T\psi_{\theta k}^{-1}[\tilde{\theta}_k-\theta_k(X,Y)]$$

Derivation of location estimates and EEP's (error ellipses) from this form is straight forward in that the old algorithms are used with the small modification that formulas involving observation covariances are divided by the probability of that observation's being true.

B. Approach 2- Use a weighted average of the original estimate and the update that would apply if the new bearing were known to be true.

The weight used is the probability that the bearing is true. If
(x_1,y_1), S_1=the estimate and covariance matrix without the new bearing
(x_2,y_2), S_2=the estimate and covariance matrix with the new bearing
$(\bar{x},\bar{y})$, $\bar{S}$ =the combined estimate and covariance

$z_1 = (x_1,y_1)$; $z_2 = (x_2,y_2)$; $\bar{z} = (\bar{x},\bar{y})$

Then for $\bar{z}=pz_1+(1-p)z_2$ it is shown in Appendix 3 that

$$\bar{S}=p[S_1+\begin{pmatrix} x_1^2 & x_1y_1 \\ x_1y_1 & y_1^2 \end{pmatrix}]+(1-p)[S_2+\begin{pmatrix} x_2^2 & x_2y_2 \\ x_2y_2 & y_2^2 \end{pmatrix}]-\begin{pmatrix} (\bar{x})^2 & \bar{x}\,\bar{y} \\ \bar{x}\,\bar{y} & (\bar{y})^2 \end{pmatrix}$$

Note: This approach is analogous to one found in the Bar-Shalom paper previously referenced.

C. Comparison of approaches 1 and 2.

Approach 2 only works well in its current form for recursive algorithms. See for example the approach 2 comments for the perpendicular method (which is not recursive) described in section IV. There are theoretical considerations which suggest that when approach 2 applies it is superior. These considerations are discussed below.

Approach 1 only accounts for one of two ellipse size considerations, however. If p=.1 for each of 10 LOBs added then approach 1 is similar to adding 9 LOBs with p=0. Approach 1 does not make any adjustment for the increase in uncertainty owing to inclusion of a wild bearing. However, approach 2 makes such an adjustment and is generally more robust (i.e. it accounts for some error types not built into the model). The covariance matrix generated by approach 2 has more ways to slow EEP shrinkage if innovations are bigger than expected. Simulation has not yet been done to check the implications of the theoretical considerations concepts discussed here.

Further investigation may reveal some method of combining these two approaches to get the advantages of both.

III. WILD BEARINGS AND SEQUENTIAL ESTIMATION

Notation is defined for what follows in the notation appendix. It will be assumed that P_1, the probability of a wild bearing, is known.

The remaining portions of the algorithm for sequential estimation will be discussed using both of the approaches of Section II.

A. Approach I - The Likelihood Modification Method

The modified objective function using approach 1 is

$$J(B|B_{k-1},\tilde{\theta}_k)=(B_{k-1}-B)^T\,\psi^{-1}_{Bk-1}(B_{k-1}-B)+(1-P_k)(\tilde{\theta}_k-\theta_k(B))^T\psi^{-1}_{\theta k}(\tilde{\theta}_k-\theta_k(B))$$

The update for the location estimate becomes

$$B_k=B_{k-1}+\{\psi^{-1}_{Bk-1}+(1-P_k)D^T_{\theta kk-1}\psi^{-1}_{\theta k}D_{\theta kk-1}\}^{-1}*(1-P_k)D^{-1}_{\theta k_{k-1}}\psi_{\theta k}[\tilde{\theta}_k-\theta_k(B_{k-1})]$$

The natural choice for the covariance of the estimate is

$$\psi^{-1}_{Bk}=\psi^{-1}_{Bk-1}+(1-P_k)D_{\theta kk}\psi^{-1}_{\theta k}D^T_{\theta kk}$$

B. Approach 2 - Weighted Location Estimate

Let $(q_k,r_k)=\{\psi_{Bk-1}^{-1}+D_{\theta kk-1}\psi_{\theta k}^{-1}D_{\theta kk-1}^T\}^{-1}D_{\theta kk-1}\psi_{\theta k}^{-1}[\tilde{\theta}_k-\theta_k(B_{k-1})]$

$B_k=B_{k-1}+(1-P_k)(q_k,r_k)$

$$\psi_{Bk}=P_k[\psi_{Bk-1}+\begin{pmatrix} x_{k-1}^2 & x_{k-1}y_{k-1} \\ x_{k-1}y_{k-1} & y_{k-1}^2 \end{pmatrix}]+(1-P_k)[(\psi_{Bk-1}^{-1}+D_{\theta kk}\psi_{\theta k}^{-1}D_{\theta kk}^T)^{-1}$$
$$+P_k(1-P_k)\begin{pmatrix} q_k^2 & q_kr_k \\ q_kr_k & r_k^2 \end{pmatrix}$$

Note the similarity of the location estimate to that used in method 1.

IV. WILD BEARINGS AND THE PERPENDICULAR METHOD

Let (u_k,v_k) denote the location of the detector of the kth LOB.

Let $\tilde{\theta}_k$=the angle measured clockwise from true north to the LOB.

Let $M_k=\begin{pmatrix} \cos^2\tilde{\theta}_k & -\sin\tilde{\theta}_k\cos\tilde{\theta}_k \\ -\sin\tilde{\theta}_k\cos\tilde{\theta}_k & \sin^2\tilde{\theta}_k \end{pmatrix}$

Let $d_k^2(x,y)=(x-u_k)^2\cos^2\tilde{\theta}_k+(y-v_k)^2\sin^2\tilde{\theta}_k-2(x-u_k)(y-v_k)\sin\tilde{\theta}_k\cos\tilde{\theta}_k$
=the square of the distance from (x,y) to the kth LOB
(measured along a perpendicular to the LOB)

It will be assumed that the probability, P_k, of the kth LOB's being wild is known. Location estimation will be examined using the two types of modification discussed in Section II.

A. Approach I - The Likelihood Modification Method

The function to be minimized without wild bearings is a variation of

$$\sum_{k=1}^N d_k^2(x,y)$$

The variations are modified weightings in this sum. The wild bearing modification is of this type. The modified likelihood is

$$\sum_{k=1}^N (1-p_k)d_k^2(x,y)$$

The resultant point estimate is at

$$(x,y)^T=[\sum_{k=1}^N(1-p_k)M_k]^{-1}\ \sum_{k=1}^N[M_k(u_k,v_k)^T]$$

B. Approach II - Weighted Location Estimate

Let (x_N,y_N) denote the Nth location estimate. Then

$$(x_N,y_N)^T=p_N(x_{N-1},y_{N-1})^T+(1-p_N)[\sum_{k=1}^N M_k]^{-1}\ \sum_{k=1}^N M_k(u_k,v_k)^T$$

The problem with this approach is most easily seen in the case where the probability of a wild bearing is sufficiently small that the probability of two wild bearings is negligible and can be dropped. After dropping the negligible terms, $P_1,\ldots P_{N-1}$ are lost. To resolve this problem it is neccessary to track at least N estimates (and ideally 2^N) depending on which estimates are wild.

V. WEIGHT (PROBABILITY) OF A PARTICULAR BEARING BEING WILD

The generation of a statistical distance as suggested in Section I consideration (2c) above requires mixing the distributions of two types of error. These two types of error are:

(1) error in one's previous estimate of the location

(2) error in observed angle

Because these errors are described in different coordinate systems it is not easy to combine them. In what follows we make the same simplifying assumption that was made in deriving the perpendicular method. That assumption is that perpendicular distance from the target to the line of bearing (LOB) can be used instead of angular error. (For a partial solution of this problem without making the perpendicular distance approximation, see the Mathematical Appendix.)

It is necessary to decide if procedures ensure that $P_1=P_2=0$, if instead $P_1=P_2=P_W$, or if some other assumption is appropriate. In fact it might be better to assume $P_i=P_W$ until it is certain that a target of some sort has been identified. After that point has been reached we assume that two proportionalities affect the weighting:

$$\text{(likelihood observed)} : \text{(likelihood of an average wild bearing)} \quad (1)$$

$$1-P_W : P_W \quad (2)$$

Evaluated and combined this yields that weights are proportioned according to

$$N(\tilde{\theta}_i-\theta(\bar{B}_{i-1});0,\psi_{Ci})*(1-P_W) : (\mu_i/L_i)*P_W$$

where N(a;b,c)=the value of the normal density with mean b, variance c, evaluated at a. The acceptance value referenced would typically be independent of index i.

From this proportionality relationship and since the sum of the two probabilities is one, we have

$$P_i=(\mu_i/L_i)*P_W/[N(\tilde{\theta}_i-\theta(\bar{B}_{i-1});0,\psi_{Ci})*(1-P_W)+(\mu_i/L_i)*P_W]$$

APPENDIX 1 - RECTANGULAR TO ANGULAR COVARIANCE CONVERSION

The likelihood which must be converted from rectangular to angular coordinates is given by

$$L(x,y)=(1/(2\pi|S|)\exp(-(x-x_0,y-y_0)S^{-1}(x-x_0,y-y_0)^T)$$ where $(x_0,y_0)=B_{i-1}$ =previous location estimate

In this appendix a density $L(\Theta)$ is computed from this density. This is done in section A below. Since the result is not normal an approximation is computed in section B using some of the same approximations that were used in derivation of the perpendicular method.

A. L(Θ) Exactly.

To express the likelihood function in polar coordinates first define

$$a=(\sin\Theta,\cos\Theta)S^{-1}(\sin\Theta,\cos\Theta)^T$$
$$b=(x_0,y_0)S^{-1}(\sin\Theta,\cos\Theta)^T$$
$$c=(x_0,y_0)S^{-1}(x_0,y_0)^T$$

The likelihood function in polar coordinates is:

$$f(r,\Theta)r\ dr\ d\Theta=(1/(2\pi|S|))\exp(-(ar^2-2br+c))r\ dr\ d\Theta$$

Elimination of r to get a likelihood depending only on Θ requires integrating this expression with respect to r. First complete the square using the following terms.

Let $R=b/a$ $D=c-aR^2$

$$g(\Theta)=(1/(2\pi|S|))\int_0^\infty \exp(-(ar^2-2br+c))r\ dr$$
$$=(1/(2\pi|S|))\int_0^\infty \exp(-a(r-R)^2-D)r\ dr$$
$$=(\exp(-D)/(2\pi|S|))\int_{-R}^\infty \exp(-a(r-R)^2)((r-R)+R)\ d(r-R)$$

There is usually no probability of r<0 so it should be a reasonable approximation to use

$$(\exp(-D)/(2\pi|S|))\int_{-\infty}^\infty \exp(-a(r-R)^2)((r-R)+R)\ d(r-R)$$

Odd functions integrate to 0 therefore the above is

$$=(\exp(-D)/(2\pi|S|))\int_{-\infty}^\infty \exp(-a(r-R)^2)R\ d(r-R)$$

$$=(\exp(-D)/(2\pi|S|))\ (\sqrt{\pi}/\sqrt{a})\ R$$

It is necessary to multiply this density by a normal and integrate from 0 to π. This is simplified somewhat by evaluation of d which turns out to be

$$[x_0\cos\Theta-y_0\sin\Theta]^2/(a|S|)$$

It is also simplified by assuming that a is a constant (if the ellipse were a cicle it would be.) Recall however that R=b/a. Thus we reduce to

$$=b(\exp(-[x_0\cos\Theta-y_0\sin\Theta]^2/(a|S|))/(2a|S|\sqrt{\pi a})$$

B. A Normal Approximation for $L(\theta)$

In this approximation the first step is to evaluate at the closest point rather than integrate over all points. First note the closest point

$(x-x_0, y-y_0) = \pm(\text{distance from LOB to location estimate})(-\cos\theta(B_i)), \sin\theta(B_i))$

$$= \pm r_i \sin(\tilde{\theta}-\theta(B_i))(-\cos(\theta(B_i)), \sin(\theta(B_i)))$$

$$= \pm r_i (\theta-\theta(B_i))(-\cos(\theta(B_i)), \sin(\theta(B_i)))$$

This form of the approximation is just what is needed to rewrite

$$(x-x_0)S^{-1}(x-x_0)^T \quad \text{as} \quad (\theta-\theta(B_i))\psi^{-1}(\theta-\theta(B_i))$$

In particular $\psi^{-1} = r_i^2(-\cos(\theta(B_i)), \sin(\theta(B_i)))S^{-1}(-\cos(\theta(B_i), \sin(\theta(B_i)))$

APPENDIX 2 - NOTATIONAL APPENDIX

Let P_W=the probability that an average bearing is wild. (Given)

P_i=the 'probability' that the ith observed bearing is wild. (Goal)

$\tilde{\theta}_i$=observed angle for ith LOB measured clockwise from North. (Observed)

B_i=estimate of target location after i bearings. (Recursive)

B_i is a vector (x_i, y_i).

$\theta_i(B_j)$=angle from detector of the ith LOB to B_j. (Computed)

r_i=distance from z_i to the detector of LOB i.

$\psi_{\theta i}$=covariance matrix for the observation $\tilde{\theta}_i$.

ψ_{Bi}=estimated covariance matrix for the estimate B_i. (Recursive)

$\psi_{Ci}=r_i^2([-\cos(\tilde{\theta}_i),\sin(\tilde{\theta}_i)]\psi_{Bi-1}^{-1}[-\cos(\tilde{\theta}_i),\sin(\tilde{\theta}_i)]^T)^{-1}+\psi_{\theta i}$

See the Appendix for a discussion of the problems in determining ψ_{Ci}. This is an approximation.

L_i=Length of the acceptance interval (in θ) determined by ψ_{Ci}.

μ_i=probability of being in the acceptance interval (of length L_i).

$D_{\theta ij}=[-\cos(\theta_i(B_j)),\sin(\theta_i(B_j))]$

APPENDIX 3 - WEIGHTED AVERAGE COVARIANCE DERIVATION

Given that:

a) An estimate $z_1=(x_1,y_1)$ would be used if the latest bearing was known to be wild.

b) An estimate $z_2=(x_2,y_2)$ would be used if the latest bearing was known not to be wild.

c) The covariance for z_1 would be S_1 (if it applied) and the covariance for z_2 would be S_2 if it applied. Furthermore,

$$S_i=\begin{pmatrix} a_i & b_i \\ b_i & c_i \end{pmatrix}$$

d) The probability that the bearing is wild is p. For notational convenience also define $p_1=p$ and $p_2=1-p$.

e) The combined estimate $\bar{z}=p_1z_1+p_2z_2=pz_1+(1-p)z_2$.

The combined estimate $\bar{z}$ has covariance $\bar{S}=$

$$=\sum_{i=1}^{2} p_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bar{z}-z)(\bar{z}-z)^T N(z;z_i,S_i)dxdy$$

$$=\sum_{i=1}^{2} p_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} (\bar{x}-x)^2 & (\bar{x}-x)(\bar{y}-y) \\ (\bar{x}-x)(\bar{y}-y) & (\bar{y}-y)^2 \end{pmatrix} N(z;z_i,S_i)dxdy$$

Thus the $\bar{a}$ entry of $\bar{S}$ is

$$\bar{a}=\sum_{i=1}^{2} p_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bar{x}-x)^2\ N(z;z_i,S_i)dxdy$$

$$=\sum_{i=1}^{2} p_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(x_i-x)^2+2xx_i-x_i^2-2x\bar{x}+(\bar{x})^2]N(z;z_i,S_i)dxdy$$

$$=\sum_{i=1}^{2} p_i\ [a_i+2x_i^2-x_i^2-2x_i\bar{x}+(\bar{x})^2]$$

$$=\sum_{i=1}^{2} p_i\ [a_i+(\bar{x}-x_i)^2]$$

$$=p_1[a_1+(\bar{x}-x_1)^2]+p_2[a_2+(\bar{x}-x_2)^2]$$

$$=p_1[a_1+x_1^2]+p_2[a_2+x_2^2]-(\bar{x})^2$$

Similar expressions are obtained for the other entries of $\bar{S}$. The combined

$$\bar{S}=p_1[S_1+\begin{pmatrix} x_1^2 & x_1y_1 \\ x_1y_1 & y_1^2 \end{pmatrix}]+p_2[S_2+\begin{pmatrix} x_2^2 & x_2y_2 \\ x_2y_2 & y_2^2 \end{pmatrix}]-\begin{pmatrix} (\bar{x})^2 & \bar{x}\,\bar{y} \\ \bar{x}\,\bar{y} & (\bar{y})^2 \end{pmatrix}$$

In some cases it might be preferable to use the equivalent form

$$\bar{S}=pS_1+(1-p)S_2+p(1-p)\begin{pmatrix} (x_1-x_2)^2 & (x_1-x_2)(y_1-y_2) \\ (x_1-x_2)(y_1-y_2) & (y_1-y_2)^2 \end{pmatrix}$$

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